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Polynomially Decaying Transmission for the Nonlinear Schrödinger Equation in a Random Medium[1]

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This is the first study of one of the transmission problems associate to the nonlinear Schrödinger equation with a random potential. We show that for almost every realization of the medium the rate of transmission vanishes when increasing the size of the medium; however, whereas it decays exponentially in the linear regime, it decays polynomially in the nonlinear one.

KEY WORDS: Localization; nonlinear waves; disordered media.

1. INTRODUCTION

In this paper we study the propagation of nonlinear waves in one-dimensional random media. Such a problem arises a priori in many fields and in particular we have in mind condensed matter physics, plasma physics, optics, and so forth. We focus here for simplicity and pedagogy on the nonlinear Schrödinger equation, but our results do apply to many other situations, as can be easily seen from the approach developed in the paper and as is sketched in the conclusion section. We will mainly study the permanent regime, but the results as discussed clearly have implications on the time-dependent problem in some time and length scale regime.

The situation for the linear case is now well-known[(1–4)] and will be recalled in Section 1: for a given incident wave with frequency ω, the transmission coefficient for a system of finite length L decays exponentially with L; this phenomenon is closely related to the phenomenon known as Anderson localization in condensed matter physics.

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The problem is a priori much more difficult for the nonlinear case, and techniques used to study the linear case seem to fail completely in the nonlinear one. Surprisingly there are, as far as we know, no result about the transmission of nonlinear waves in random media, at least if the backscattering is present (a situation in which the backscattering is not present has been discussed in Rosales and Papanicolaou, Ref. 5). Thus we want to know how the exponential decay of the transmission may have been modified by the nonlinearities or, in other words, we want to know how localization theory has been modified by the presence of nonlinearities. This problem is also nontrivial from the numerical point of view since many approaches are numerically unstable.

In fact, as we will see, the situation is very different in the nonlinear case from the linear one: first of all, as we will see, the transmission problem is no more uniquely defined and we will need to make more precise the problem under discussion. Then, although the transmission of an incident wave will again tend to zero as the size of the system increases, and although the decay will still be exponential when the wave is in the linear regime, when nonlinearities do play a role, the decay is much slower, essentially L^{-1}.

In Section 1, we pose more precisely the problem and in particular we recall the situation for the linear case.

In Section 2, we discuss the nonlinear case and make precise which transmission problem we are looking at. We first prove that the transmission does go to zero as the length L of the system goes to infinity but we also prove that as long as nonlinearities are relevant the transmission cannot decay faster than L^{-2}. Finally we present numerical and theoretical results indicating that in fact it decays as L^{-1} in this regime.

2. STATEMENT OF THE PROBLEM: THE LINEAR CASE

In the present section, we recall how to study the transmission coefficient for the linear case. The eigenvalue equation for a linear Schrödinger equation with a random potential $V(x)$ is

$$-\Psi_{xx} + V(x)\Psi = k^2\Psi \tag{1}$$

The precise form of the random potential is certainly not essential in our problem, and so we restrict ourselves to the simplest model, namely that V is a stepwise constant function: in each interval $[x, x+s]$, V is constant and chosen with some probability to be equal to $V_{\min}$ or $V_{\max}$, the length of each step being chosen randomly on $[0, +\infty[$ with a density; the Vs on different steps are assumed to be independent random variables. The case where $k^2 \leqslant V_{\max}$ is trivial and thus we will always restrict ourselves to the case where $k^2 \geqslant V_{\max}$.

Therefore, we must study the stationary Schrödinger equation. When an incident plane wave of wavelength k attacks a slab of disordered medium of length L, we get a reflected wave and a transmitted one, of respective wavelengths k and $-k$. From the theory of wave propagation in disordered media,[1-4] it is known that, for almost every realization of the medium [that is, of the potential $V(x)$], the transmission $|t_1/r_0|^2$ of the slab tends asymptotically exponentially to zero as L tends to infinity: $|t_1/r_0|^2 \sim \exp(-2L/\xi)$ as $L \to \infty$ where ξ is a characteristic length. This phenomenon is closely related to a phenomenon well-known in condensed matter physics under the name of Anderson localization, which predicts that in some conditions of disorder and, in particular, for any disorder in dimension one, all the proper modes of the Schrödinger equation with a random potential are exponentially localized for almost every realization of the random potential; for reviews on localization theory see, for example Refs. 7 and 8 and see Ref. 9 for a large bibliography. The connection between the exponential decay of the transmission and the localization phenomenon for disordered systems is clarified in Ref. 10.

The condition expressing that we have an outgoing wave at the right of the sample is

$$\Psi(x) = t_1 e^{ikx} \qquad \text{for} \quad x > 0 \tag{2}$$

On the left of the sample we have

$$\Psi(x) = r_0 e^{ikx} + r_1 e^{-ikx} \qquad \text{for} \quad x < -L \tag{3}$$

and the conserved current reads

$$J = (\Psi^* \Psi_x - \Psi \Psi_x^*)/2i = |t_1|^2 = |r_0|^2 - |r_1|^2 \tag{4}$$

The method used to show that the transmission tends to zero exponentially as L tends to infinity uses theorems on the product of random matrices: on the nth step the solution can be written as

$$\Psi(x) = a_n e^{ikx} + b_n e^{-ikx} \tag{5}$$

and if we denote by M_n the transfer matrix on the nth step, that is, the 2×2 matrix which allows us to compute a_n and b_n from a_{n+1} and b_{n+1}, we have

$$(t_1, 0)^t = M(L) \cdot (r_0, r_1)^t \tag{6}$$

where $M(L) = M_0 M_{-1}, ..., M_{-L}$. Equation (6) can now be rewritten as

$$t_1^{-1} \cdot (r_0, r_1)^t = M(L)^{-1} \cdot (1, 0)^t \tag{7}$$

and a well-known theorem of Fürstenberg[11] can be applied (for a review of the mathematical results on products of random matrices we refer the reader to Ref. 12. It tells us that for given k and for almost all realizations of the medium the vector obtained from the fixed vector $(1, 0)^t$ by applying the random matrix $M(L)^{-1}$ is exponentially increasing with L, from which it readily follows in view of Ref. 7 that the transmitted power $|t_1/r_0|^2$ decreases exponentially with L.

3. POWER LAW DECAY OF THE TRANSMISSION IN THE NONLINEAR REGIME

We turn now to the study of the nonlinear case for which the approach used for the linear case, and recalled above, clearly cannot be transposed in a straightforward way. As already mentioned, we restrict ourselves for simplicity to the study of the nonlinear Schrödinger equation with a random potential although some of our results can be easily extended to many other situations.

We thus consider the nonlinear Schrödinger equation for a slab of nonlinear disordered medium of length L, outside of which the medium is supposed for simplicity to be linear and nonrandom (see Fig. 1). Since, as will appear below, we will be mainly concerned with the stationary equation and since for $\alpha < 0$, its solutions blow up at a finite distance x, we consider here only the case where $\alpha > 0$. Thus we study the equation

$$i\Phi_t = \Phi_{xx} + \alpha(x)\,|\Phi|^2\,\Phi - V(x)\Phi \tag{8}$$

with

$$\begin{aligned} \alpha(x) &= \alpha > 0 && \text{for} \quad -L \leqslant x \leqslant 0 \\ &= 0 && \text{for} \quad x < -L \text{ and } x > 0 \\ V(x) &= 0 && \text{for} \quad x < -L \text{ and } x > 0 \end{aligned}$$

and $V(x)$ on $-L \leqslant x \leqslant 0$ is the same random potential as in Section 1.

Since we study only the permanent regimes we look for solutions of (8) of the type

$$\Phi(x, t) = e^{ik^2 t}\Psi(x)$$

so that we have to study the stationary equation

$$-\Psi_{xx} - \alpha(x)\,|\Psi|^2\,\Psi + V(x)\Psi = k^2\Psi \tag{9}$$

and as in Section 1 we restrict ourselves only to nontrivial situations where $k^2 > V_{\max}$.

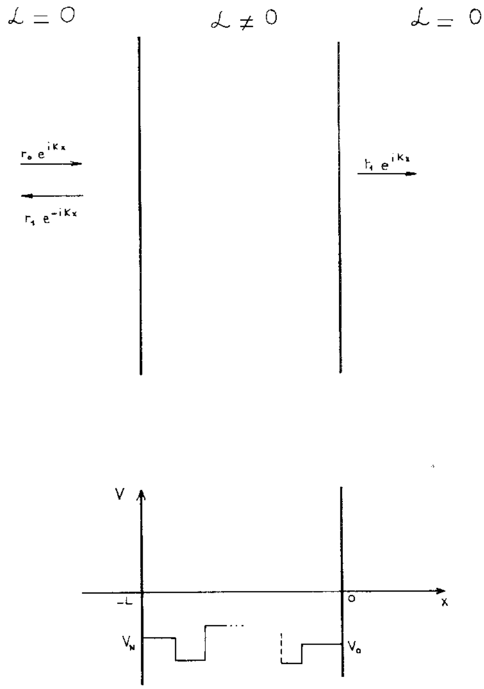


Fig. 1. The situation studied in the present paper: a standing wave in a piece of nonlinear and disordered material, between two pieces of linear and homogeneous material. On the right of the material for $x \geqslant 0$ the wave is only outgoing, whereas on the left for $x \leqslant -L$ there is an incoming and a reflected wave.

Note that in contrast to the linear case the general time-dependent problem is no longer completely reducible to the study of the stationary problem. In particular we have neglected the creation of harmonics; at this point it may be nevertheless interesting to notice that the localization length of higher harmonics is larger than the one for the fundamental, and thus coupling to the harmonics should contribute to a delocalization. In a straightforward way, the restrictions above can be turned into length and time scale conditions for the applicability of the results of the present work to physical situations.

We want to study the transmission problem associated to the stationary nonlinear Schrödinger equation (9); it corresponds as in Section 1 to the conditions

$$\Psi(x) = t_1 e^{ikx} \qquad \text{for} \quad x > 0 \tag{10}$$

$$\Psi(x) = r_0 e^{ikx} + r_1 e^{-ikx} \qquad \text{for} \quad x < -L \tag{11}$$

We note that the current

$$J = (\Psi^* \Psi_x - \Psi \Psi_x^*)/2i = |t_1|^2 = |r_0|^2 - |r_1|^2$$

is still conserved.

But the first question we have to ask ourselves is the following: What is really the transmission problem we want to look at? This question has to be asked now since in contrast to the linear case it is no longer equivalent to study the transmission problem for fixed input r_0 or fixed output t_1. In this paper we study the question of the transmission for fixed output which arises, for instance, if one needs a fixed minimal power at the end of the slab in order to activate some device, and we thus ask for the corresponding necessary input power as a function of the length of the slab.

We note that for given output $|t_1|^2$, that is, for given current J, there is one and only one solution of the problem given by (9–11). (In contrast for fixed input amplitude r_0, there is always at least one solution (t_1, r_1) to the problem but, because of nonlinearities, there may be more than one for a given L, and interesting results on bistability phenomenon in that case have been obtained recently.[13] Thus we can now ask for the behavior of the transmission $|t_1/r_0|^2$ as a function of the length L of the system. The first question to ask is certainly whether the transmission still goes to zero as the length L of the sample goes to infinity; we will answer this question and prove that indeed this is the case. The next question is then, of course, how does it goes to zero? For small enough t_1 the problem will be almost linear, at least as long as L is not too large, and we thus should expect an exponential decay of transmission in this regime. We will obtain it, but in contrast we show that, when the nonlinearity begins to exert itself, that is, for larger t_1 or for any t_1 and sufficiently large L, then the transmission behaves in a polynomial way and in fact does not decay faster than L^{-2}. Since we do not get by this method the exact law of the behavior in this regime we then study this question numerically and theoretically and we argue that the exact behavior in the nonlinear regime is of the type L^{-1}.

Let us state now our exact results more precisely, the numerical and theoretical ones being discussed later.

Theorem. (1) The transmission $|t_1/r_0|^{2+\varepsilon}$ goes to zero as $L \to \infty$ in the sense of the Cesaro mean, for any $\varepsilon > 0$, for almost every realization of the medium.

(2) For almost every realization of the medium, the transmission $|t_1/r_0|^2$ behaves exponentially for J small enough and L not too large.

(3) For any J the transmission $|t_1/r_0|^2$ cannot tend toward zero faster than L^{-2} as $L \to \infty$.

In order to study the decrease of the transmission coefficient and obtain result (1), we develop a method introduced in Ref. 14 for the study of the linear case: this method was somehow forgotten since it yields only a weak form of decay in contrast to the exponential decay obtained easily using the Fürstenberg theorem; however, as we see now, it can be used to apply to nonlinear cases. The result of (2) concerning the exponential behavior as long as the linear regime is dominant will follow from the fact that the Lyapunov exponent of some nonlinear transformation can be proven to be nonzero in some neighborhood of zero, thus yielding the existence of stable and unstable manifolds in this neighborhood and allowing the use of standard techniques of dynamical systems theory. Finally, general a priori bounds will provide us with a lower bound on the decay of the transmission, ensuring result (3).

(a) Proof of Part 1 of the Theorem. Following Ref. 14, we introduce the impedance $z = \Psi_x/\Psi$ instead of the wave function Ψ (note that $\Psi = 0$ would imply $J = 0$ and thus z is always finite). Equation (9) thus becomes the Ricatti equation

$$z_x + z^2 + k^2 - V(x) + \alpha J/\mathrm{Im}(z) = 0 \qquad \text{for} \quad -L \leqslant x \leqslant 0 \tag{12}$$

and the boundary conditions (10) and (11) become

$$z(0) = i \tag{13}$$

$$r_1/r_0 = [i - z(-L)]/[i + z(-L)] \tag{14}$$

The equations for the integration of $z = u + iv$ on a step of constant V are

$$u_x = -u^2 + v^2 - (k^2 - V) - \alpha J/v \tag{15}$$

$$v_x = -2uv \tag{16}$$

and Ψ, Ψ_x, and thus z are continuous at the edge of the steps.

If we denote by $\Phi_{s,V}$ the integration of (12)–(14) from the right to the left end of a step of length s and potential V, we have, for a slab of N steps of total length $L = \sum_{1 \leqslant i \leqslant N} s_i$

$$z(-L) = \Phi^N[z(0)] = \Phi^N(i) \tag{17}$$

where Φ^N holds for $\Phi_{s_N, V_N} \circ \cdots \circ \Phi_{s_2, V_2} \circ \Phi_{s_1, V_1}$ and thus r_1 can be readily computed as

$$r_1(N) = r_0(N) \cdot [i - \Phi^N(i)]/[i + \Phi^N(i)] \tag{18}$$

Let us introduce the function $f(z)=(1-|i-z|/|i+z|)^{1+\varepsilon}$ where ε is an arbitrarily small positive number. The Cesaro mean of $(1-|r_1|/|r_0|)^{1+\varepsilon}$ is defined by

$$N^{-1}\sum_{1\leqslant n\leqslant N}[1-|r_1(n)/r_0(n)|]^{1+\varepsilon}=N^{-1}\sum_{1\leqslant n\leqslant N}f[\Phi^n(i)]$$

and we introduce

$$F(z)=\lim_{N\to\infty}N^{-1}\sum_{0\leqslant n\leqslant N-1}f[\Phi^n(z)]$$

when it exists.

At this step it is important to note that all the transformations Φ^n conserve the measure $d\mu=du\cdot dv/v^2$: it suffices to set $w=1/v$ and to check that the volume is conserved in the u, w plane; namely, $du\cdot dw$ is conserved because

$$(\partial/\partial u)\,u_x+(\partial/\partial w)\,w_x=0$$

and in fact the term due to the nonlinearity in (15) plays no role in it because $(\partial/\partial u)(\alpha J/v)\equiv 0$.

In order to simplify notation, we denote by a_j the randomness on the jth step (from the right); that is, the length s_j of the step and the height V_j of the potential on it: $a_j\equiv(s_j, V_j)$. A configuration of potential on $x\leqslant 0$ will be denoted $\omega\equiv\{a_j\}_{j\in\mathbb{N}}$ and $\Omega\equiv\{\omega\}$ denotes the set of possible configurations of the medium. The probability distribution on the possible configurations is denoted by $\mathbb{P}$. We introduce the "skew product" G from $(\Omega\times\mathbb{C}_+)$ into itself

$$G(\omega, z)=[\tau(\omega), \Phi_{a_1}(z)]$$

$\mathbb{C}_+$ being the upper complex half plane $v>0$ and τ denoting the translation on ω, that is, $\tau(\{a_i\})\equiv\{a_{i+1}\}$; that Φ sends $\mathbb{C}_+$ into itself can be seen directly from (15) and (16).

As we see in Appendix 1, the application G is ergodic on $\Omega\times\mathbb{C}_+$ with respect to the measure $\mathbb{P}\times\mu$. We thus suppose from now on that G is ergodic (note, however, that this is a nontrivial property, as will appear in the Appendix, and this is the place where the specific properties of the equation that we study do enter; it is the property which may not be true for some other equations). It thus follows from the random ergodic theorem[15] that $F(z)$ exists and is equal to a constant C for $(\mathbb{P}\times\mu)$-almost every (ω, z) of $\Omega\times\mathbb{C}_+$. Since $f\in L^1(\mathbb{C}_+; d\mu)$ and since $\int_{\mathbb{C}_+}d\mu=\infty$, the measure $d\mu$ has infinite mass on $\mathbb{C}_+$ and we can induce that the constant C

has to be 0. In particular we thus have that for almost every realization of the medium, $F(z)$ exists and is equal to zero for μ-almost every z of $\mathbb{C}_+$.

Since we are interested in the transmission problem we are interested—for a given typical medium—specifically in $F(i)$; however i could belong to the set of those z with zero μ-measure for which the result of the previous paragraph would not hold. In order to solve this point we need to remark that the above result is in fact an asymptotic one and can thus be rephrased as: "for $\mathbb{P}$-almost every $\{a_i\}_{i\geqslant 3}$, then for every a_1, a_2, and z such that $\Phi_{a_2}\circ\Phi_{a_1}(z)$ does not belong to some fixed set of $\mathbb{C}_+$ with zero μ-measure, $F(z)$ exists and is equal to zero." It is easy to check, as we do in Appendix 2, that $\Phi_{a_2}\circ\Phi_{a_1}$ is nondegenerate with respect to a_1 and a_2 in the sense that the set of $\Phi_{a_2}\circ\Phi_{a_1}(i)$ is of nonzero Lebesgue measure when s_1 and s_2 are varied continuously. It thus follows that for $\mathbb{P}$-almost every $\{a_i\}_{i\geqslant 1}$, that is, $\mathbb{P}$-almost every realization of the medium, $F(i)$ exists and is equal to zero. This ensures the convergence to zero of the transmission as announced in result 1 of the Theorem. ∎

(b) Proof of Part 2 of the Theorem. Let us consider the nonlinear transformation G on $\Omega\times\mathbb{C}_+$ that we introduced above; if the parameter α of the nonlinearity is set equal to zero (this is the linear case), this transformation has two (opposite) nonzero Lyapunov exponents; using general perturbative results[17,18] we can deduce that G for small enough α has again two opposite nonzero Lyapunov exponents; this tells us that for fixed α, but small enough current J, the transformation G has two opposite nonzero Lyapunov exponents. In turn it follows the existence of a stable and an unstable manifold for G in a neighborhood of zero for $\mathbb{P}$-almost every realization ω of the medium. The announced result then follows from this fact through standard techniques that we thus skip. ∎

(c) Proof of Part 3 of the Theorem. On each step with constant potential V_n, the nonlinear Schrödinger equation (9) conserves the quantity

$$H_n = |\Psi_x|^2 + (k^2 - V_n)\,|\Psi|^2 + \alpha\,|\Psi|^4/2 \tag{19}$$

Since Ψ and Ψ_x are continuous at the edges of the steps, we have

$$H_{n+1} - H_n = (V_n - V_{n+1})\,|\Psi(n)|^2$$

and thus

$$|H_{n+1} - H_n| \leqslant (\Delta V)\,|\Psi(n)|^2 \tag{20}$$

where ΔV denotes the maximum allowed fluctuation of the potential that is $\Delta V = V_{\max} - V_{\min}$. From (20), using (19), we obtain

$$|H_{n+1} - H_n| \leqslant (\Delta V)(2H_n/\alpha)^{1/2} \tag{21}$$

and from (21) one proves by induction that

$$H_n \leqslant (2\alpha)^{-1}(\Delta V)^2(n+A)^2 \tag{22}$$

for some A; note that the dependence with respect to J lies only in A. When the nonlinear medium has N steps, one can bound from below H_N as a function of the transmission:

$$\begin{aligned} H_N &= k^2(r_0 e^{ikL} - r_1 e^{-ikL})^2 + (k^2 - V_N)(r_0 e^{ikL} + r_1 e^{-ikL})^2 \\ &\quad + (\alpha/2)(r_0 e^{ikL} + r_1 e^{-ikL})^4 \qquad (23) \\ &\geqslant (k^2 - V_N)(|r_0|^2 + |r_1|^2) \\ &= (k^2 - V_N)(2\,|r_0|^2 - J) \qquad (24) \end{aligned}$$

and the announced result follows by comparison of (22) and (24). (Note that almost surely the total length of a system of N steps is proportional to N). ∎

Theoretical and Numerical Improvements of the Results

We have seen above that due to nonlinearities, the transmission $|t_1/r_0|^2$ cannot decay fast for long systems and in fact it cannot decay faster than L^{-2}. Since this result was only a bound, we would like to discuss now if this rate is the exact one or if the transmission decays still slower. For this it is natural to first look to some numerical results before going back to the theoretical analysis: the solutions $\Phi_{s,V}$ of (9)–(11) on a step of constant potential V and length s can be explicitly calculated in terms of the Jacobi elliptic functions, from which we can deduce numerically the value of $\Phi^N(i)$ and thus of the transmission.

We performed numerically the product $\Phi^N(i)$ for a given current J and different values of α; more precisely, since the only relevant parameter is αJ, we have made computations for various values of this parameter. For αJ small ($\alpha J = 10^{-13}$) the transmission, calculated from $\Phi^N(i)$, first decays exponentially when N increases as predicted in statement 2 of the Theorem above, and then slower. The crossover region appears approximately when $\alpha J/v$ is of order $k^2 - V$, which is natural (see Fig. 2). For αJ large ($\alpha J = 1$), we went up to 7000 iterations; the ratio $|t_1/r_0|$ has very important fluctuations and does not allow an average behavior to be extracted but the Cesaro mean of $1 - |r_1/r_0|$ decays as L^{-1} for large N, thus corresponding to a decay of the transmission $|t_1/r_0|^2$ as L^{-1}, a decay apparently slower than the bound obtained in statement 3 of the Theorem above (see Fig. 3).

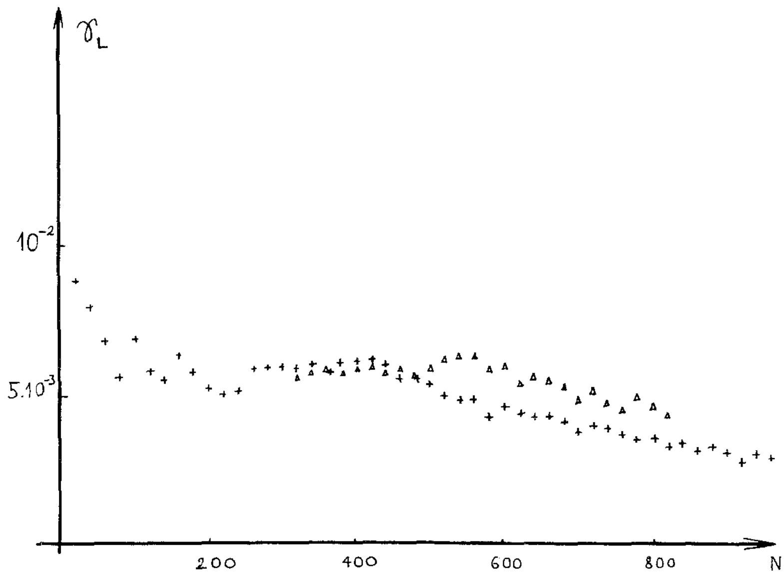


Fig. 2. The regime of exponential decay of the transmission for small current (or small nonlinearity) and short enough length; γ_L stands for $L^{-1} \log |\Psi(-L)/\Psi(0)|$. The + corresponds to the case $\alpha J = 10^{-13}$, the Δ to $\alpha J = 10^{-20}$. In both cases, we have $s_{\min} = 5$, $s_{\max} = 11$, $n_{\min} = (k^2 - V_{\max})^{1/2} = 1.2$, $n_{\max} = (k^2 - V_{\min})^{1/2} = 4.4$.

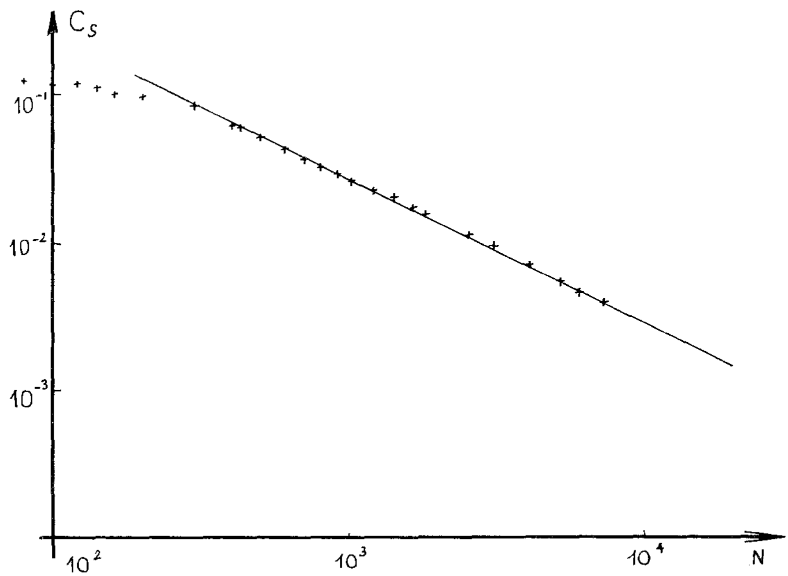


Fig. 3. The regime of slow polynomial behavior for large current or long systems: C_L stands for the Cesaro sum $N^{-1} \sum (1 - |r_1/r_0|)$. Here $\alpha J = 1$, $s_{\min} = 5$, $s_{\max} = 11$, $n_{\min} = (k^2 - V_{\max})^{1/2} = 1.2$, $n_{\max} = (k^2 - V_{\min})^{1/2} = 12$.

Can we understand theoretically this true behavior of the transmission? Yes, in the following way: let's go back to (23). Instead of going to the lower bound (24) which is obtained only from the two first terms of (23), we could use the third term of (23) yielding

$$H_N \geqslant (\alpha/2)[|r_0|^2 + |r_1|^2 + 2\,\mathrm{Re}(r_0 r_1^* e^{2ikL})]^2 \tag{25}$$

from which together with (22) and $|r_0|^2 - |r_1|^2 = J$ would result that the transmission $|t_1/r_0|^2$ does not decay faster than L^{-1}; this is valid except when the argument of $r_0 r_1^* e^{2ikL}$ is near to π mod(2π). In this latter case this new contribution is small and we only recover the bound associated to the first two terms of (23). However, the phase is turning and will not remain near π; we thus obtain from (25) an upper bound on the transmission $|t_1/r_0|^2$ of the type L^{-1} better than the one of statement 3 of the Theorem. This argument could be made rigorous. On the other hand, an "adiabatic theory" using the behavior of the Lyapunov exponent for the linear Schrödinger equation at large energy can be used to obtain estimations in the other direction. It suggests that the true behavior may be of type L^{-1} for the case of a steplike potential like the one we use here but should be even slower for smoother potentials.

4. CONCLUSION: DISCUSSION AND EXTENSIONS

We have studied the transmission for a class of nonlinear wave equations. For the problem we have defined (as we have seen, the transmission problem is not uniquely defined for the case of nonlinear equations in contrast to the linear case) we have first shown that the transmission vanishes for long systems for almost every realization of the medium; it is clear from the proof that the same result holds for a very large class of distributions of potentials and is not at all restricted to the distribution we have choosen for the sake of simplicity in the present paper. More interestingly the proof also extends to a very large class of other nonlinear equations, much beyond the special case of the nonlinear Schrödinger equation. Since this extension is easy from our computations we decided to restrict ourselves to a pedagogical example.

We then showed that in the almost linear regime, that is, for small current and small length, the transmission is exponentially decaying with length; again this result is very general and extends to very large classes of distributions and of nonlinear equations since in fact this is basically a perturbative result and it depends almost only on whether the linear equation has an exponentially decaying transmission.

Finally we have shown that in the nonlinear regime, that is, for large current or long enough systems, the transmission cannot decay exponentially but in fact only very slowly. It has to be noted that such a result is in fact a deterministic one and thus does not depend on the distribution of the potential, and many other potentials can be accommodated in this type of argument; however, it depends crucially on the nonlinearity of the equation; the restriction on the decay shown in the present paper is due to the strong nonlinearity of the nonlinear Schrödinger equation. Analogous results can hold for other types of equations but the precise dependence will depend strongly on the equation and on the a priori bounds that one can get on the solutions of the Cauchy problem associated to it.

APPENDIX 1: SUFFICIENT CONDITIONS FOR THE ERGODICITY OF *G*

In this Appendix we intend to prove that the transformation G is ergodic on $\Omega \times \mathbb{C}_+$ with respect to the measure $\mathbb{P} \times \mu$ for the random potential studied in this paper; from the proof below it is clear that ergodicity can be proven for a very large class of ergodic random potentials on $\mathbb{R}$.

In order to show the ergodicity of G, it is sufficient[16] to show that the family $\{\Phi_{s,V}\}$ is ergodic in the sense that the only subsets of $\mathbb{C}_+$, invariant for the family, have zero or full measure with respect to μ [a subset B of $\mathbb{C}_+$ is invariant for the family if for $\mathbb{P}$-almost every (s, V), the symmetrical difference of B and $\Phi_{s,V}(B)$ has zero μ-measure].

It follows, as can be verified, that in our case it is sufficient that, given two arbitrary points of $\mathbb{C}_+$, we can find a realization of the potential such that they can be connected by Φ transformations, i.e.

$$\forall (z_1, z_2) \in \mathbb{C}_+ \times \mathbb{C}_+, \qquad \exists (\omega, N) \in \Omega \times \mathbb{N} \text{ s.t. } z_1 = \Phi^N(z_2)$$

We are going to show now this property for the random potential we have choosen: in order to simplify notations, and since k^2 here is fixed, we introduce

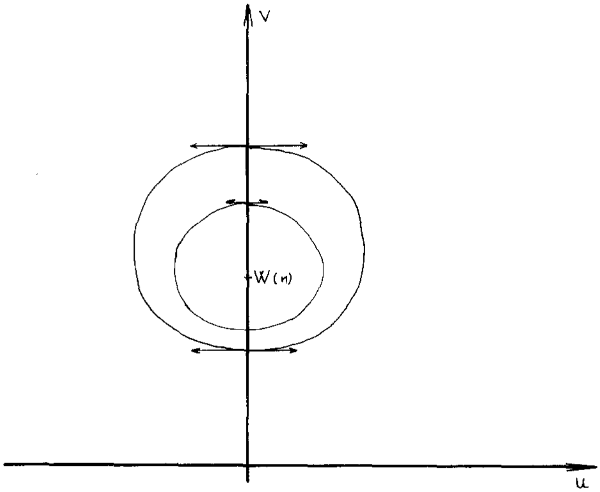
$$n_1 = (k^2 - V_{\max})^{1/2} \qquad \text{and} \qquad n_2 = (k^2 - V_{\min})^{1/2}$$

the applications $\Phi_{s,V}$ being correspondingly relabeled $\Phi_{n,s}$.

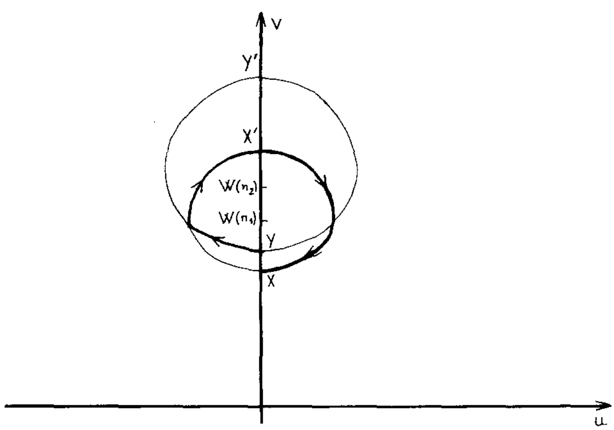
For a given constant potential V, that is, for n constant, there is a family of curves $C(n, r)$ of $\mathbb{C}_+$ which are invariant by the applications $\Phi_{n,s}$ for all s and which are given by the equations

$$u^2 + [v - (n^2 + r)^{1/2}]^2 + \alpha J/2v = r$$

for any $r \in [r_{\min}, +\infty[$, for some $r_{\min} > 0$. This family of invariant curves fills $\mathbb{C}_+$ entirely. Each of these invariant curves is contained in a rectangle,

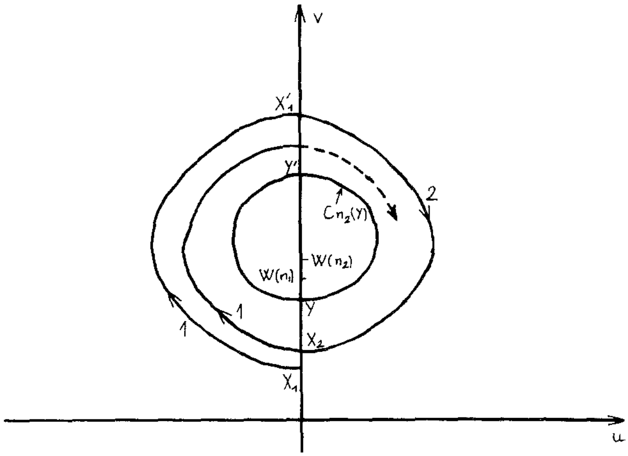


(a)



(b)

Fig. 4. (a) Two of the curves $C(n, r)$ invariant by $\Phi_{n,s}$ for all s; here, in the case $\alpha J = 100$, $n = (k^2 - V)^{1/2} = 1.5$. If $r_2 > r_1 > r_{\min}$, $C(n, r_1)$ is contained in $C(n, r_2)$. (b) How to connect X and Y: here is the simplest case when X' the $\mathrm{Conj}_{n_1}(X)$ is below Y' the $\mathrm{Conj}_{n_2}(Y)$. The picture is drawn with $\alpha J = 100$, $n_1 = 1.2$, $n_2 = 1.5$. (c) The construction of the "spiral." The 1 and 2 indicate, respectively, the arcs of curves $C(n_1, r)$ and $C(n_2, r)$.



(c)

Fig. 4 *(continued)*

and a line parallel to one of the coordinate axis can intersect in $\mathbb{C}_+$ the curve at most at two points (see Fig. 4a). On the other hand, two curves of the family for the same n but of different r never intersect. Finally we note that for each n there is a point $W(n)$ on the y axis, with ordinate $w(n)$, which lies in the interior of all curves of the family; this point is itself the invariant curve obtained for $r=r_{\min}$; we note that $w(n)$ is a strictly increasing function of n.

Now on each curve $C(n, r)$, two arbitrary points can be connected by at least one transformation $\Phi(n, s)$ for some s, since we have supposed for simplicity that the support of the distribution of s is $\mathbb{R}^+$. On the other hand, to every point P of $\mathbb{C}_+$ is associated a curve $C_{n_1}(P)$ belonging to the family of curves $C(n_1, r)$ invariant by $\Phi_{n_1,s}$ for all s; the point P can thus be connected by a $\Phi_{n_1,s}$ transformation to the point of $C_{n_1}(P)$ located on the y axis below $W(n_1)$ and so it is now sufficient for us to show that two arbitrary points on the y axis, say X and Y, $X<Y$, lying between the origin and $W(n_1)$, can be connected.

In order to achieve this, let us denote by $\mathrm{Conj}_{n_2}(\cdot)$ the conjugation by a n_2 curve, that is, the application which to any given point of the positive y axis associates the other point of the positive y axis which belongs to the same $C(n_2, r)$ curve. We define similarly $\mathrm{Conj}_{n_1}(\cdot)$ the conjugation by an n_1 curve. Now if X', the $\mathrm{Conj}_{n_1}(X)$ lies below Y' the $\mathrm{Conj}_{n_2}(Y)$, then the $C(n_1, r)$ curve passing through X and X' intersects the $C(n_2, r)$ curve passing through Y and Y'. We are then in the situation of Fig. 4b and X and Y are connected. If this is not the case, we define the sequence of points (see Fig. 4c): $X_1=X$, $X_m'=\mathrm{Conj}_{n_1}(X_m)$, $X_{m+1}=\mathrm{Conj}_{n_2}(X_m')$ and we consider the spiral-like continuous curve consisting of the pieces of $C(n_1, r)$ curves between points X_m, X'_m and the pieces of $C(n_2, r)$ curves between points X'_m, X_{m+1}. If this "spiral" meets $C_{n_2}(Y)$, the $C(n_2, r)$ curve passing through the point Y, then we can connect X and Y through Φ transformations. We now show that this has to be the case.

Suppose that this spiral never meets the $C_{n_2}(Y)$ curve. It means that $X'_i > Y'$ and $X_i < Y$ always. But the sequence of X'_i would be a strictly decreasing sequence, provided $W(n_1) \neq W(n_2)$ (i.e., provided $n_1 \neq n_2$). It thus has a limit point $X'_{\lim}$ with $X'_{\lim} \geq Y'$ and $X'_{\lim} = \mathrm{Conj}_{n_2} \circ \mathrm{Conj}_{n_1}(X'_{\lim})$, and since Conj_{n_2} is an involution we would have $\mathrm{Conj}_{n_2}(X'_{\lim}) = \mathrm{Conj}_{n_1}(X'_{\lim})$. If $n_1 \neq n_2$, this last equality is impossible to satisfy for a point $X'_{\lim}$ of ordinate higher than Y', and, thus, than $W(n_2)$.

APPENDIX 2: NONDEGENERACY OF $\Phi_{n,s}$

It is sufficient to prove that $\mu(A) > 0$ where A is the set defined by

$$A = \{\Phi_{n_a,s_a} \circ \Phi_{n_b,s_b}(i);\ (n_a, n_b) \in \{n_1, n_2\}^2,\ (s_a, s_b) \in \mathbb{R}^+ \times \mathbb{R}^+\}$$

and n_1, n_2 are as in Appendix 1. Let us set

$$A_0 = \{\Phi_{n_b,s_b}(i);\ n_b \in \{n_1, n_2\},\ s_b \in \mathbb{R}^+\}$$

and denote by I the point of affix i and by P the $\mathrm{Conj}_{n_1}(I)$ where the conjugation has the same meaning as in Appendix 1. The set A_0 is made of the

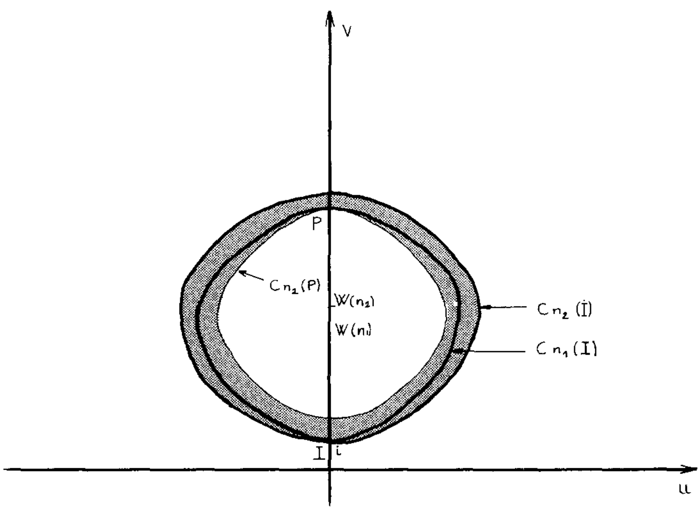


Fig. 5. The set $\{\Phi_{n_2,s_2}(x);\ s_2 \in \mathbb{R}^+,\ x \in C_{n_1}(I)\}$.

two curves $C_{n_1}(I)$ and $C_{n_2}(I)$ as defined in Appendix 1. Now the ensemble $\{\Phi_{n_2,s_2}(x); s_2 \in]0, +\infty[, x \in C_{n_1}(I)\}$ will in fact be the domain of the complex plane lying between $C_{n_1}(I)$ and $C_{n_2}(P)$, because the value of the parameter r for which the curve $C_{n_2}(x)$ is a curve $C(n_2, r)$ as defined in Appendix 1 varies continuously and monotonously as x goes from I to P on $C_{n_1}(I)$. This ensemble, the shaded area on Fig. 5, is contained in A and has nonzero μ-measure as soon as $n_1 = n_2$. ▮

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NOTE ADDED IN PROOF

Concerning the influence of nonlinearities on Anderson localization, see also the paper by J. Fröhlich, T. Spencer and C. Wayne, to appear in *J. Stat. Phys.*

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